Proportional power is free from paradoxes *

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Abstract

We modify the story behind the Shapley-Shubik power index and apply it to a legislative body. The resulting proportional index may be trivial, but is free from the paradoxical behaviour observable with standard power indices. The widespread use of this index may in fact be the reason for these "paradoxes".
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1 Introduction

The reform of the EU legislative mechanisms have put weighted voting and voting power in particular in the spotlight. Since Shapley and Shubik (1954) applied the Shapley-value to simple games numerous indices have been introduced and the search for the best index is far from over: all the known indices exhibit some or more of the so-called paradoxes (Brams, 1975/2003): true statements that are absurd (Felsenthal and Machover, 1998).

While these paradoxes are often dismissed as mere features, the selection from the plethora of power indices remains difficult. Axiomatisation seemed to be the right way to organise indices, but axioms are not less ad hoc than the indices themselves (Laruelle and Valenciano, 2005, pp37-38). Also rejecting the (complete) axiomatisation approach, Felsenthal and Machover (1995) suggest a set of desirable properties, but so far no index satisfies their *postulates*.

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We approach the problem from both ends. On the one hand we present a motivation for the proportional, a new, trivial power index, On the other we argue that the paradoxes may be a result of this naive intuition about power. As scientific models enter popular thinking we should also regard 'paradoxes' outdated or no more than 'apparently strange pieces of behaviour' (Felsenthal and Machover, 1998, p. 221)

The structure of the paper is as follows. After the introduction of basic concepts and notation we define the proportional index. Then we elaborate on the paradoxes and show that this trivial index is immune to the paradoxes. We close with a discussion of this result's implications.

2 The 'trivial' index is free from paradoxes

A *weighted voting game* $\mathrm{G} = (N, (w_i)_{i \in N}, q)$ consists of a collection N of n *voters* having $w_1, w_2, \ldots, w_n$ votes such that $w = \sum_{i=1}^{n} w_i$, and a quota q, $w \geq q > w/2$, or the number of votes *required* to pass a bill. For more on weighted voting games see Straffin (1994). A power index is a function k that assigns to each weighted voting game a non-negative vector in $\mathbb{R}_{+}^{N}$.

The *proportional index* α is the trivial power index given by $\alpha_i = \frac{w_i}{w}$.

This measure is popularly known in political science as Gamson's Law: 'Any participant will expect others to demand from a coalition a share of the payoff proportional to the amount of resources which they contribute' (Gamson, 1961).

Simplicity is not without merit: Brams (1975/2003) lists three natural properties that any power index should satisfy (the list is extended by Felsenthal and Machover (1998)), but the best-know indices satisfy none of these. These disappointing negative results (known as paradoxes) do not extend to the proportional index. In the following we state these properties and show that they hold for the proportional index.

Property of (large) size Let $\mathrm{G} = (N, (w_i)_{i \in N}, q)$ be a voting game and k a power index. Define G' by the merger of players i and j. The resulting party ij has a weight $w_{ij} = w_i + w_j$. The property requires $k_i(\mathrm{G}) + k_j(\mathrm{G}) \leq k_{ij}(\mathrm{G}')$.

Proposition 1. *The index α satisfies the Property of (large) size.*

Proof. Using the notation in the definition, the merged member's weight is simply $w_{ij} = w_i + w_j$, the merger does not alter the total weight, hence $\alpha_{ij}(\mathrm{G}') = \frac{w_{ij}}{w} = \frac{w_i + w_j}{w} = \alpha_i(\mathrm{G}) + \alpha_j(\mathrm{G})$. □

Property of new members Now define G$''$ as an extension of G by parties $n+1, \ldots, m$ and weighs $w_{n+1}, \ldots, w_m$ and a q'' to meet the requirements. The property requires $k_i(\mathrm{G}) \geq k_i(\mathrm{G}'')$, that is, the introduction of new members should not increase a party's power.

Proposition 2. *The index α satisfies the Property of new members.*

Proof. An 'old' member's weight remains $w'_i = w_i$, while the total increases to $w' > w$ due to the new members, hence $\alpha_i(\mathrm{G}') = \frac{w_i}{w} = \frac{w'_i}{w} > \frac{w'_i}{w'} = \alpha_i(\mathrm{G})$. □

Before we turn to Brams's last paradox we mention three listed by Felsenthal and Machover (1998) that hold for the proportional index by definition.

Property of redistribution Let $\mathrm{G} = (N, (w_i)_{i \in N}, q)$ and $\mathrm{G}' = (N, (w'_i)_{i \in N}, q)$ be two weighted voting games. The property requires that for all $i \in N$ we have $k_i(\mathrm{G}) < k_i(\mathrm{G}')$ if and only if $w_i < w'_i$.

Donation Property is then the special case where $w'_i = w_i + \delta$ and $w'_j = w_j - \delta$ for some $i, j \in N$ and $w'_k = w_k$ for all $k \notin \{i, j\}$.

Dominance Property A power index satisfies dominance in a weighted voting game G if $w_i < w_j$ implies $k_i(\mathrm{G}) < k_j(\mathrm{G})$.

Property of quarrelling members If two parties refuse to vote together, this should not increase their total power.

The property of quarrelling members has been criticised for looking at actions (quarrelling) beyond the specification of the game. Indeed, a quarrelling coalition is losing even if it meets the quota. Before we move on to the proof of this last property we present the proportional index as a modification of the Shapley-Shubik index; this approach will then be used in the proof.

Power indices have been used in legislative voting, (van Deemen and Rusinowska, 2003) where voting weights are given by the number of representatives in each group.[1] In the following we consider such a voting situation.

In the model of Shapley and Shubik (1954) voters arrive in a random order, until a pivotal player turns a losing coalition into a winning one. The Shapley-Shubik index is then the proportion of orders where the player is pivotal, formally: $\phi_i = \frac{\#\text{ times } i \text{ is pivotal}}{n!}$

A weighted voting game consists of a number of players, having different weights – the EU Council of Ministers is the classical example where countries of different sizes are represented by single, weighted votes. Such examples

[1] Results extend to irrational w_i using limits.

are relatively rare; in most national parliaments the weight of the different parties comes from having different numbers of representatives, and the power is the collection of these individual voters' power. In such a model a party is pivotal, if one of its representatives is pivotal. Formally,

$$\hat{\phi}_i = \frac{\#\text{ times a representative of } i \text{ is pivotal}}{w!}. \tag{1}$$

Now recall that a coalition is winning if it has size q or greater. Therefore a player is pivotal and gets credit if he or she is the qth to vote for. A party's power index is the probability that the qth voter is one of its representatives. However, this probability is the same as the probability that the first or any kth voter is its representative: $\frac{w_i}{w}$, the proportional index. This yields the following result:

Proposition 3. *For a voting game $(N, (w_i)_{i\in N}, q)$ we have $\hat{\phi}_i = \frac{w_i}{w} = \alpha_i$, that is, the party's share of the votes.*

Proposition 4. *The index α satisfies the Property of quarrelling members.*

Proof. When two players quarrel, coalitions containing both are losing even if they meet the quota in size.

In the following we calculate the powers of players i and j in case they quarrel. Observe that a winning coalition has always size q. Now suppose that all we know is that player i has a representatives in the coalition. Then the conditional probability that i gets credit is simply $\frac{a}{q}$. We can therefore write i's power as

$$\alpha_i(\mathrm{G}) = \frac{\sum_{a=0}^{w_i} \frac{a}{q}\binom{q}{a}\binom{w-q}{w_i-a} w_i!(w-w_i)!}{w!} = \frac{w_i}{w}. \tag{2}$$

Similarly we can condition on the number b of representatives of j in q and calculate powers. The probability that i or j win having, respectively, a and b representatives in the winning coalition is

$$\frac{a+b}{q}\binom{q}{a,b}\binom{w-q}{w_i-a, w_j-b} w_i! w_j! (w-w_i-w_j)!$$

In case i and j quarrel either a or b must be 0 (hence $ab = 0$). After some trivial simplifications the inequality to be shown is the following:

$$\alpha_i(\mathrm{G}''') + \alpha_j(\mathrm{G}''') = \tag{3}$$

$$\frac{\sum_{a=0}^{w_i}\sum_{b=0 ab=0}^{w_j} \frac{a+b}{q}\binom{q}{a,b}\binom{w-q}{w_i-a,w_j-b}}{\sum_{a=0}^{w_i}\sum_{b=0 ab=0}^{w_j}\binom{q}{a,b}\binom{w-q}{w_i-a,w_j-b}} < \frac{w_i+w_j}{w} \tag{4}$$

$$= \alpha_i(\mathrm{G}) + \alpha_j(\mathrm{G}) \tag{5}$$

Equivalent transformations lead to

$$w_i + w_j < w_i \frac{\binom{w-w_i-1}{q-1}}{\binom{w-w_i-w_j-1}{q-1}} + w_j \frac{\binom{w-w_j-1}{q-1}}{\binom{w-w_i-w_j-1}{q-1}}. \tag{6}$$

This inequality holds term-by-term. □

3 Discussion

It is remarkable that such a trivial and –admittedly– imperfect index satisfies all of Brams's properties. On the other hand, Diermeier and Merlo (2004), Gelman, Katz, and Bafumi (2004) Fréchette, Kagel, and Morelli (2005) (and references therein), find that the proportional distribution of power behaves surprisingly well both in empirical tests and coalition formation models, suggesting that assigning power proportionally is a common practice if not the standard.

The properties of this natural index are then natural, too, making anything different, such as power indices that take a more educated look at voting unnatural, even paradoxical. In this light it is not so surprising to see Luxemburg to be a null voter in the 1958 EU Council of Ministers and many would have argued it had too many votes for its size. The question that remains is whether science should formalise our intuition (stick to paradoxes) or try to change our intuition and make us accept these paradoxes as normal (that is: stick to power indices). The recent discussions about and general interest in weighted voting and power indices in connection to the EU reform suggest that we are ready for plan B.

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